Quantized-output feedback model reference control of
discrete-time linear systems

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Abstract

This paper proposes a model reference control (MRC) scheme for general discrete-time linear time-invariant (LTI) systems subject to output quantization and saturation. In such a scheme, the quantized-output feedback MRC law is analytically specified only by using external reference input and saturated-and-quantized output. It is proven that, by appropriately designing the sensitivity of the quantizer, the MRC law can ensure that all closed-loop signals are bounded and the output tracking error converges to a certain small residual set in a certain finite time only under the minimum-phase condition. Particularly, the proposed MRC scheme does not rely on the system controllable, observable, or initial conditions that are commonly used in the related literature. A representative example is given to demonstrate the design procedure and verify the validity of the proposed MRC scheme.

Key words: Model reference control, discrete-time, output tracking, quantized-output feedback.

1 Introduction

Over the past decades, we have witnessed a fast growing interest in how to effectively control linear and nonlinear systems subject to quantized and saturated measurements. On one hand, traditional state or output feedback control methods are generically not able to be directly used for systems subject to quantized and/or saturated measurements. Thus, it is of importance to systematically develop quantized and saturated control theory. On the other hand, in real control systems, it often needs to overcome the restrictions of quantization and saturation problems, especially in networked control systems and digital control systems. Comparing to traditional control methods, quantized feedback control methods greatly reduce the requirement for sensor accuracy and magnitude. Moreover, in view of anti-noises/disturbances, quantized feedback control has stronger robustness than exact feedback control. In a word, it is of greatly theoretical and practical importance to address quantized feedback control systems.

So far, quantized feedback control theory and applications have gained long-term developments, and a lot of remarkable results have been published. Quantized control research can be traced back to the 1960s (Larson (1967)). From then on, various quantized control methods have been published (Baijieul (1999); Wong and Brockett (1999); Nair and Evans (2000); Tatikonda and Mitter (2004); Elia and Mitter (2001); Fu and Xie (2005); Gao and Chen (2008); Ishii and Francis (2003); Phat et al. (2004); Yan et al. (2019); Hespanha et al. (2002); Li and Baijieul (2004); Hayakawa et al. (2009)). Quantized control idea was also widely used in multi-agent consensus or formation
systems only under the minimum-phase condition, i.e., the polynomial $B(z)$ is stable. In particular, the system observable or controllable condition, commonly used in existing quantized-output feedback control methods, is no longer needed in this paper. In summary, the contributions of this paper are as follows.

(i) A quantized-output feedback MRC scheme is developed for a general class of discrete-time LTI systems. We show that a saturated and quantized version of the standard MRC law is valid for control of the above systems.

(ii) In comparison with existing literature, the proposed MRC law has its distinctive characteristics. First, only using external reference input and saturated-and-quantized output, the MRC law is analytically constructed. Second, only under the minimum phase condition, the MRC law can ensure that all closed-loop signals are bounded and the output tracking error converges to a certain residual set in a certain finite time. Last but not the least, the MRC law is independent of the system initial conditions.

(iii) The validity of the proposed MRC scheme is verified by a representative example with simulation results.

The rest of this paper is organized as follows. Section II gives the system model and control problems. Section III reviews some fundamentals of output feedback MRC theory, based on which Section IV as the main section of this paper presents the design details of the quantized-output MRC scheme. Sections V and VI give the simulation study and concluding remarks, respectively.

2 Problem statement

This section presents the system model and the problems to be addressed in this paper.

System model. Consider the following discrete-time single-input and single-output (SISO) LTI system:

$$A(z)y(t) = k_p B(z)u(t), \quad t \geq t_0,$$

where $t_0$ is the initial time of the system operation, $z$ is the forward shift operator, $k_p \neq 0$ is the high-frequency gain, $A(z)$ and $B(z)$ are monic polynomials with constant coefficients and of degrees of $n$ and $m$. It is well-known that the traditional MRC law is of the form $u(t) = \theta_1^T \phi_1(t) + \theta_2^T \phi_2(t) + \theta_r^T r(t)$, where $\theta_1, \theta_2, \theta_r$ are constant parameters of appropriate dimensions, $\phi_1(t), \phi_2(t)$ are well-defined signals only depending on $y(t)$ and $u(t)$, and $r(t)$ is an external reference input signal (Tao 2003, Tao 2014). Up to now, it is still unclear whether a saturated-and-quantized output feedback version of the traditional MRC law is still effective. Such a problem is never addressed before in the literature. In this paper, we for the first time give a positive answer. Based on the output feedback MRC theory in Tao 2003, we establish a basic quantized-output feedback MRC framework for discrete-time LTI systems only under the minimum-phase condition, i.e., the polynomial $B(z)$ is stable. In particular, the system observable or controllable condition, commonly used in existing quantized-output feedback control methods, is no longer needed in this paper. In summary, the contributions of this paper are as follows.

(i) A quantized-output feedback MRC scheme is developed for a general class of discrete-time LTI systems. We show that a saturated and quantized version of the standard MRC law is valid for control of the above systems.

(ii) In comparison with existing literature, the proposed MRC law has its distinctive characteristics. First, only using external reference input and saturated-and-quantized output, the MRC law is analytically constructed. Second, only under the minimum phase condition, the MRC law can ensure that all closed-loop signals are bounded and the output tracking error converges to a certain residual set in a certain finite time. Last but not the least, the MRC law is independent of the system initial conditions.

(iii) The validity of the proposed MRC scheme is verified by a representative example with simulation results.

The rest of this paper is organized as follows. Section II gives the system model and control problems. Section III reviews some fundamentals of output feedback MRC theory, based on which Section IV as the main section of this paper presents the design details of the quantized-output MRC scheme. Sections V and VI give the simulation study and concluding remarks, respectively.
Dynamic quantizer. Let $X \in \mathbb{R}$ denote any signal on $\mathbb{R}$. Then, we specify the quantizer in this paper, which is the same with that in Brockett and Liberzon (2000) and has the form

$$ q(X) = \begin{cases} M, & \text{if } X > (M + \frac{1}{2})\Delta, \\ \left\lceil \frac{X}{\Delta} \right\rceil \frac{M}{2}, & \text{if } -(M + \frac{1}{2})\Delta < X \leq (M + \frac{1}{2})\Delta, \\ -M, & \text{if } X \leq -(M + \frac{1}{2})\Delta, \end{cases} \quad (4) $$

where $\Delta \neq 0$ depends on $t$ and is called the sensitivity of $q$, $M \in \mathbb{R}$ is a constant, and $[X] \triangleq \max\{k \in \mathbb{Z} : k < X\}$.

As stated in Brockett and Liberzon (2000), the form (4) of the quantizer has some certain physical meanings and potential applications, such as vision-based control. One may refer to Brockett and Liberzon (2000) to see a full clarification for the quantizer (4). The authors in Brockett and Liberzon (2000) proposed (4) to address the MRC problem that covers the stabilizability and potential applications, such as vision-based control.

Reference output model. Let $Y(t) = G(z)[U](t)$ denote an output $Y(t) \in \mathbb{R}^p$ of a plant with a transfer function $G(z) \in \mathbb{R}^{p \times q}$ and an input $U(t) \in \mathbb{R}^q$. The reference output is

$$ y^*(t) = W_m(z)[r](t), \quad W_m(z) = \frac{1}{P_m(z)}, \quad (5) $$

where $P_m(z)$ is a stable monic polynomial of degree $n - m$ and $r(t) \in \mathbb{R}$ is an external reference input signal such that $r(t) \in L^\infty$. For discrete-time MRC, it is common to choose $P_m(z) = z^{n-m}$ so that

$$ y^*(t) = r(t - n + m). \quad (6) $$

Control objective. For any given $y^*(t) \in L^\infty$, the control objective is to develop a quantized-output feedback control law $u(t)$ for the system (1) to ensure that closed-loop signals are bounded and $y(t) - y^*(t)$ converges to a certain small residual set in a certain finite time.

Assumption. To meet the control objective, we only need the following assumption.

(A1): The polynomial $B(z)$ is stable.

Assumption (A1) is equivalent to that the system (1) is minimum-phase, which is needed for analyzing internal signal boundedness (Tao (2003)). Assumption (A1) implies that the state-space form of (1) is stabilizable and detectable, but may not be controllable or observable.

3 Fundamentals of output feedback MRC

We review some fundamentals of output feedback MRC of discrete-time LTI systems in Tao (2003), which will be used in quantized-output feedback MRC design.

Matching equation. Before giving the MRC law, we first present the following lemma which specifies a key equation for the control law design.

Lemma 1. Tao (2003) There exist constant vectors $\theta_1^* \in \mathbb{R}^{n-1}$ and $\theta_2^* \in \mathbb{R}^n$ such that

$$ \theta_1^* T \omega_1(z) A(z) + k_p \theta_2^* T \omega_2(z) B(z) = A(z) - B(z) z^{n-m}, \quad (7) $$

where $\omega_1(z) = [z^{-n+1}, ..., z^-1]^T$ and $\omega_2(z) = [z^{-n+1}, ..., z^-1]^T$ with $z^{-1}$ being the backward shift operator.

The proof of this lemma can be seen in Tao (2003). The equation (7) is the well-known matching equation for output feedback MRC of LTI systems (Tao (2003)).

Output feedback MRC law. With $\theta_1^*$ and $\theta_2^*$ in (7), the MRC law is designed as

$$ u(t) = \theta_1^* T \phi_1(t) + \theta_2^* T \phi_2(t) + \frac{1}{k_p} r(t), \quad t \geq t_0, \quad (8) $$

with

$$ \phi_1(t) = \omega_1(z) u(t), \quad \phi_2(t) = \omega_2(z) y(t). \quad (9) $$

The following lemma specifies the capability of the MRC law (8).

Lemma 2. Tao (2003) There exist unique $\theta_1^*$ and $\theta_2^*$ that meet (7) and guarantee that the MRC law (8) leads to closed-loop stability and exact output tracking

$$ y(t + n - m) - y^*(t + n - m) = 0, \quad \forall t \geq t_0, \quad (10) $$

for the system (1).

The proof of Lemma 2 can be seen in Tao (2003). The parameters $\theta_1^*$ and $\theta_2^*$ in Lemma 2 are the so-called the matching parameters because with these parameters the control law (9) leads to exact matching of the closed-loop system to the reference model (5).

Remark 3. Note that if $A(z)$ and $B(z)$ are not coprime, $\theta_1^*$ and $\theta_2^*$ satisfying (7) are not unique. However, Lemma 2 indicates that, no matter whether $A(z)$ and $B(z)$ are coprime or not, the parameters $\theta_1^*$ and $\theta_2^*$ in (8) are unique. This characteristic is proven in Tao (2003) and also can be concluded from the proof of Lemma 4 in the sequel. The proof of Lemma 4 also specifies how to determine the unique parameters $\theta_1^*$ and $\theta_2^*$ especially...
for the case when $A(z)$ and $B(z)$ are not coprime. By the way, if the control law (8) uses an arbitrary pair of $\theta_1^*$ and $\theta_2^*$ satisfying (7), it can only lead to the tracking error as $B(z)e(t+n-m) = 0$ which cannot imply exact tracking if not all zeros of $B(z)$ are at the origin. □

Lemmas 1-2 are the fundamentals of MRC of discrete-time LTI systems and also the foundation of the quantized-output feedback MRC scheme that will be systematically addressed in the sequel.

4 Quantized-output feedback control design

Based on Lemmas 1-2, this section develops a quantized-output feedback MRC scheme for the system (1).

4.1 Quantized-output feedback MRC law structure

The standard output feedback MRC law (8) motivates us to design the quantized-output feedback MRC law of the following structure

$$u(t) = \theta_1^T \phi_1(t) + \theta_2^T \phi_q(t) + \frac{1}{k_p} r(t), \quad t \geq t_0, \quad (11)$$

where $\theta_1^*, \theta_2^*$ are the unique parameters in Lemma 2, and

$$\phi_q(t) = \omega_2(z)(\Delta(t)q(y(t))) \quad (12)$$

with $\Delta(t)$ being the sensitivity of the quantizer $q$ in (4) to be designed later.

Tracking error equation. Define the quantized error and the tracking error as

$$s(y(t)) = \Delta(t)q(y(t)) - y(t), \quad e(t) = y(t) - y^*(t), \quad (13)$$

respectively. Now, we give the following lemma which specifies the tracking error equation that will be crucial for the sensitivity $\Delta(t)$ design and stability analysis.

**Lemma 4** The quantized-output MRC law (11), applied to the system (1), ensures

$$e(t+n-m) = k_p \theta_2^T \omega_2(z) s(y(t)), \quad \forall \ t \geq t_0. \quad (14)$$

**Proof.** From (7), we have

$$(\theta_1^T \omega_1(z) - 1)A(z) = (-k_p \theta_2^T \omega_2(z) - z^{-n-m}) B(z). \quad (15)$$

We first consider the case when $A(z)$ and $B(z)$ are coprime. It follows from (15) that, if $z_i$ is a zero of $B(z)$, it must be a zero of $\theta_1^T \omega_1(z) - 1$, otherwise (15) does not hold for $z = z_i$ with $B(z)$ and $A(z)$ coprime. Thus, we conclude that there exists some polynomial

$$F(z) = -z^{-m} + f_{n-m-2} z^{-m-1} + \cdots + f_{0} z^{-n+1} \quad (16)$$

with $f_i, i = 0, ..., n - m - 2$, being constant coefficients such that

$$F(z)B(z) = \theta_1^T \omega_1(z) - 1. \quad (17)$$

In addition to (15), we obtain

$$k_p \theta_2^T \omega_2(z) + F(z)A(z) = -z^{-m}. \quad (18)$$

Then, operating both sides of (18) on $y(t)$ yields

$$k_p \theta_2^T \omega_2(z)y(t) + F(z)A(z)y(t) = -y(t+n-m). \quad (19)$$

Together with (1) and (9), it further implies

$$k_p \theta_2^T \phi_2(t) + k_pF(z)B(z)u(t) = -y(t+n-m). \quad (20)$$

Substituting (17) to (20) derives

$$k_p \theta_2^T \phi_2(t) + k_p(\theta_1^T \omega_1(z) - 1)u(t) = -y(t+n-m). \quad (21)$$

Using (9) again, we get

$$k_p \theta_2^T \phi_2(t) + k_p \theta_1^T \phi_1(t) - k_p u(t) = -y(t+n-m). \quad (22)$$

Substituting the quantized-output feedback law (11) to (22), together with (6), we obtain

$$y(t+n-m) - y^*(t+n-m) = k_p \theta_2^T(\phi_q(t) - \phi_2(t)) = k_p \theta_2^T \omega_2(t)s(y(t)). \quad (23)$$

Now we consider the case when $A(z)$ and $B(z)$ are not coprime. In this case, we rewrite $B(z)$ as $B(z) = B_1(z)B_2(z)$ such that $B_1(z)$ has degree $n_1$, and $B_2(z)$ and $A(z)$ are coprime. Then, there exist unique parameters $\theta_1^* \in \mathbb{R}^{n-1-n_1}$ and $\theta_2^* \in \mathbb{R}^n$ such that

$$z^{n-1-n_1} \theta_1^T \omega_1(z)A(z) + k_p \theta_2^T z^{-1-n_1} \omega_2(z)B_2(z) = z^{n-1-n_1} A(z) - z^{-n} B_2(z) z^{-m} \quad (24)$$

with $\bar{\omega}_1(z) = [z^{-n+n_1+1}, ..., z^{-1}]^T$. With some manipulations, (24) becomes

$$z^{-n_1} \theta_1^T \omega_1(z)A(z) + k_p \theta_2^T \omega_2(z)B_2(z) = z^{-n_1} A(z) - B_2(z) z^{-n-m} \quad (25)$$

Similar to (17), there exists some polynomial of the form

$$F(z) = -z^{-m+n_1} + f_{n-m+n_1-2} z^{-m+n_1-1} + \cdots + f_{0} z^{-n+n_1+1} \quad (26)$$
such that \( \bar{F}(z)B_2(z) = \theta_1^T\bar{z}_1(z) - 1 \). Let \( F(z) = z^{-m_1}F(z) \). Then, in addition to (25), we also obtain (18), based on which the lemma’s result follows. Note that, for the non-coprime case, the parameter \( \theta_1^T \) is uniquely determined from the equation \( \theta_1^T\bar{z}_1(z) - 1 = (\theta_1^T\bar{z}_1(z) - 1)B_2(z)z^{-m_1} = F(z)B(z)\).

The tracking error equation (14) implies that exact output feedback, i.e., \( s(y(t)) = 0 \), can achieve exact output tracking. However, for the quantized-output feedback case, before designing \( \Delta(t) \), it cannot be sure whether \( s(y(t)) \) is bounded or not. Thus, the tracking error equation (14) does not imply the boundedness of \( e(t) \). Next, we will show that, with an appropriate choice of \( \Delta(t) \), \( s(x(t)) \) can be made arbitrarily small, which follows from (14) that \( c(t) \) can be made arbitrarily small.

### 4.2 Three technical lemmas

To proceed, we give the following three lemmas that will be helpful to design \( \Delta(t) \). Define

\[
\lambda \triangleq \text{an upper bound of } \max\{1 + \text{magnitudes of } \lambda_i(A(z))\},
\]

where \( \lambda_i(A(z)) \), \( i = 1, 2, ..., n \), denote the zeros of \( A(z) \) on the complex \( z \)-plane.

Now, we give the following lemma.

**Lemma 5** For the system (1), if \( u(t) = 0 \) and \( \Delta(t) = c_0\lambda^kt \) with \( c_0 > 0 \) and \( k > 1 \) being any two constants, then there always exists a well-defined number \( t_1 \) as

\[
t_1 \triangleq \min\{t_1 \geq t_0 + 1 : |q(y(t))| \leq M - 1\}.
\]

**Proof.** If \( u(t) = 0 \), \( y(t) \) grows at most exponentially and \( |y(t)| \leq k_0(\lambda - 1)^t \) with \( k_0 \) being some constant. If \( \Delta(t) = c_0\lambda^kt \) with \( c_0 > 0 \) and \( k > 1 \), \( \Delta(t) \) grows faster than \( |y(t)| \). Thus, no matter whether \( q(y(t_0)) \) saturates or not, there exists some finite time instant \( t_p \) such that \( q(y(t)) \) will never saturate for all \( t \geq t_p \). Then, the lemma’s result follows from the definition of \( q \) in (4). \( \nabla \)

To design \( \Delta(t) \), we also need the following lemmas.

**Lemma 6** For any given constants \( \gamma \in \mathbb{R} \) and \( N \in \mathbb{Z} \) such that \( 0 < \gamma < 1 \) and \( N > 1 \), if \( M \) is large enough, then the inequality holds:

\[
M - \frac{1}{2} \geq c + \frac{d}{c_0\gamma^{N-1}\lambda^k(t_1+n-m-1)}, \tag{29}
\]

where \( c_0 > 0 \) is a constant, and

\[
c \triangleq \frac{1}{2}\|k_p\|\theta_2^T\|1, \quad d \triangleq \text{an upper bound of } |y^*(t)|. \tag{30}
\]

**Lemma 7** If \( |y(t)| \leq (M - \frac{1}{2})\Delta(t) \), then

\[
|s(y(t))| \leq \frac{1}{2}\Delta(t). \tag{31}
\]

The proofs of Lemma 6 and Lemma 7 are not difficult to perform. We omit them for space. Next, with Lemmas 5-7, we systematically address how to specify \( \Delta(t) \).

#### 4.3 Control design for systems with \( n - m = 1 \)

To show the basic ideas, we first consider the relative degree one case. Then, we address the general case.

**Quantized-output MRC law for relative degree one case.** For the system (1) with \( n - m = 1 \), the quantized-output MRC law is designed as

\[
u(t) = \begin{cases} 
0, & t \in [t_0, t_1), \\
\theta_1^T\phi_1(t) + \frac{1}{c_{p_r}}r(t) + \theta_2^T\omega_2(z)(\Delta(t)q(y(t))), & t \in [t_i, t_{i+1}), \quad i = 1, ..., N - 1, \\
\theta_1^T\phi_1(t) + \frac{1}{c_{p_r}}r(t) + \theta_2^T\omega_2(z)(\Delta(t)q(y(t))), & t \in [t_N, \infty),
\end{cases} \tag{32}
\]

where \( \Delta(t) = c_0\lambda^kt \) for \( t \in [t_0, t_1) \),

\[
\Delta(t_i) = c_0\gamma^{i-1}\lambda^k, \quad i = 1, 2, ..., N, \tag{33}
\]

\( c_0, k, \gamma \) are chosen constants such that \( c_0 > 0 \), \( k > 1 \) and \( 0 < \gamma < 1 \), \( \lambda \) in (27), \( t_1 \) in (28), and \( t_i, i = 2, 3, \cdots, N \), as

\[
t_i \triangleq \min\left\{t_i \geq t_{i-1} + 1 : |q(y(t))| \leq \frac{c_0\Delta(t_{i-1}) + |y^*(t)|}{\Delta(t_{i-1}) + \frac{1}{2}} \right\}. \tag{34}
\]

with \( c \) in (30).

**System performance analysis.** With the MRC law (32), we derive one of the main results as follows.

**Theorem 8** Under Assumption (A1), if \( M \) satisfies (29), then the quantized-output MRC law (32), applied to the system (1) with \( n - m = 1 \) and any unmeasurable \( y(t_0) \) \( \in \mathbb{R} \), ensures that all closed-loop signals are bounded and the tracking error satisfies

\[
|e(t)| \leq c_0\lambda^k\gamma^{N-1}, \quad \forall t \geq t_N + 1. \tag{35}
\]

**Proof.** For any unmeasurable \( y(t_0) \) \( \in \mathbb{R} \), Lemma 5 ensures the existence of \( t_1 \). When \( t = t_1 \), from Lemma 4, we
have \(|y(t)| \leq \Delta(t_1) \left(M - \frac{1}{2}\right)\). Moreover, when \(t = t_1\), we change the control law from \(u(t) = 0\) to (11) with \(\Delta(t) = \Delta(t_1)\). Then, from Lemma 4, we have

\[
y(t_1 + 1) - y^*(t_1 + 1) = kp\theta_1^T \omega_2(z)s(y(t_1)).
\]  

(36)

Since \(q\) does not saturate at \(t = t_1\), Lemma 7 implies

\[
|s(y(t_1))| \leq \frac{1}{2} \Delta(t_1).
\]

(37)

Moreover, based on the fact that \(\Delta(t) < \Delta(t_1)\) for all \(t < t_1\), combining (36) and (37) yields

\[
|y(t_1 + 1) - y^*(t_1 + 1)| = \left|\frac{1}{2} kp\theta_1^T \omega_2(z)\Delta(t_1)\right| \leq c\Delta(t_1). \tag{38}
\]

Then, we have

\[
|y(t_1 + 1)| \leq c\Delta(t_1) + |y^*(t_1 + 1)|. \tag{39}
\]

Thus, from (39) and Lemma 6, we obtain

\[
|y(t_1 + 1)| \leq \Delta(t_1) \left(M - \frac{1}{2}\right) \tag{40}
\]

which implies that \(q(y(t_1 + 1))\) also does not saturate. Then, we can further verify that \(q(y(t_1 + 2))\) does not saturate, neither do \(q(y(t_1 + 3)), q(y(t_1 + 4)), \ldots\). Therefore, we conclude that, if the control law is chosen as (11) for \(t \geq t_1\), \(q(y)\) will never saturate. Moreover, \(e(t)\) satisfies that

\[
e(t) \leq c\Delta(t_1), \quad \forall t \geq t_1 + 1. \tag{41}
\]

Thus, there exists a well-defined number \(t_2\) as

\[
t_2 \triangleq \min \left\{ t \geq t_1 + 1 : |q(y(t))| \leq \frac{\Delta(t_1) + |y^*(t)|}{\Delta(t_1)} + \frac{1}{2} \right\}. \tag{42}
\]

Then, when \(t = t_2\), we change the control law to (11) with \(\Delta(t) = \gamma \Delta(t_1)\). Then,

\[
|y(t_2)| \leq \Delta(t_1) \left(|y(t_2)| + \frac{1}{2}\right) \leq c\Delta(t_1) + \Delta(t_1) \leq \Delta(t_2) \left(M - \frac{1}{2}\right) \tag{43}
\]

which implies that \(q(y(t_2))\) does not saturate. Thus, using Lemma 7 again, we obtain

\[
|y(t_2 + 1) - y^*(t_2 + 1)| \leq c\Delta(t_2), \tag{44}
\]

\[
|y(t_2 + 1)| \leq c\Delta(t_2) + |y^*(t_2 + 1)|. \tag{45}
\]

Together with Lemma 6, if \(N \geq 2\), we have

\[
|y(t_2 + 1)| \leq \Delta(t_2) \left(M - \frac{1}{2}\right) \tag{46}
\]

which implies that \(q(y(t_2 + 1))\) does not saturate. Then, we can also verify that \(q(y(t_2 + 2))\) does not saturate, neither do \(q(y(t_2 + 3)), q(y(t_2 + 4)), \ldots\). Therefore, we conclude that, if the control law is chosen as (11) with \(\Delta(t) = \gamma \Delta(t_1)\) for \(t \geq t_2\), \(q(y)\) will never saturate and \(e(t)\) satisfies that

\[
e(t) \leq \gamma c\Delta(t_1), \quad \forall t \geq t_2 + 1. \tag{47}
\]

Repeating the above procedure, we define \(t_3, t_4, \ldots, t_N\) and obtain a sequence \(\{\Delta(t_i)\}_{1 \leq i \leq N}\). Moreover, we see that the control law (32) ensures that \(e(t)\) satisfies (35). By \(y^*(t) \in L^\infty\), we have \(y(t) \in L^\infty\). Under Assumption (A1), the boundedness of \(y(t)\) implies that of \(u(t)\). $\n$

Based on (29) and (35), Theorem 8 indicates that if \(M\) is larger, \(N\) can be chosen larger and it follows from (35) that \(e(t)\) can be made smaller.

**Remark 9** For the sake of explanation, \(t_{i+1}\) in (34) can be roughly understood as the time when \(e(t)\) first reaches the set \(\{e(t) : |e(t)| \leq c\Delta(t_i)\}\). In the proof of Theorem 8, we have shown that the proposed MRC law (32) with \(\Delta(t) = \Delta(t_i)\) and \(t \geq t_i\) ensures that \(e(t)\) reaches the above set in a finite time and remain inside thereafter. Thus, \(t_{i+1}\) must be existing and finite. However, we cannot use \(e(t)\) to define \(t_i\) as \(e(t)\) is not available. Instead, we use available signals \(\Delta(t_i)\) and \(y^*(t)\) to define \(t_{i+1}\) in (34). The finite time convergence of \(e(t)\) guarantees that \(t_{i+1}\) in (34) is also existing and finite. $\square$

### 4.4 Control design for systems with an arbitrary relative degree

Now, we give the following theorem that shows that the quantized-output MRC law (32) is also effective for the system (1) with \(n - m > 1\).

**Theorem 10** Under Assumption (A1), if \(M\) satisfies (29), then the quantized-output MRC law (32) with \(\Delta(t) = c_0 \lambda^kt\) for \(t \in [t_0, t_1]\) and

\[
\Delta(t_i) = c_0 \gamma^{i-1} \lambda^{k(t_1 + n + m - 1)}, \quad i = 1, 2, \ldots, N, \tag{48}
\]

with \(t_1\) in (28) and \(t_i\) in (34), applied to the system (1) with any unmeasurable \(y(t_0) \in \mathbb{R}\) and \(1 \leq n - m \leq n\), ensures that all closed-loop signals are bounded and

\[
e(t) \leq c_0 \lambda^{k(t_1 + n + m - 1)} N^{n-1}, \quad \forall t \geq t_N + n - m. \tag{49}
\]
Proof. Similar to the $n - m = 1$ case, for any unmeasurable $y(t_0) \in \mathbb{R}$, Lemma 5 ensures the existence of $t_1$. When $t = t_1$, it follows from Lemma 4 that

$$|y(t_1)| \leq c_0\lambda t_1 (M - \frac{1}{2}).$$  \hspace{1cm} (50)

For $t \geq t_1$, we change the control law from $u(t) = 0$ to (32) with $\Delta(t) = \Delta(t_1)$ in (48). In particular, when we change the control law from $t = t_1$, due to the input-output delay $n - m$, the control law (32) does not influence $y(t_1 + j)$, $j = 1, 2, ..., n - m - 1$. Thus, based on the fact that $c_0\lambda t$ grows faster than $y(t)$, we have

$$|y(t_1 + j)| \leq c_0\lambda^{t_1+t_j} (M - \frac{1}{2}) \leq \Delta(t_1) (M - \frac{1}{2})$$ \hspace{1cm} (51)

for $j = 0, 1, ..., n - m - 1$ and $\Delta(t_1) = c_0\lambda^{t_1+n-m-1}$.

Then, (51) implies that $y(t_1 + j)$, $j = 0, 1, ..., n - m - 1$, all do not saturate. Moreover, for $j = 0, 1, ..., n - m - 1$, from Lemma 4, we have

$$y(t_1 + n - m + j) - y^*(t_1 + n - m + j) = k_p\theta_2^T w_2(z)s(y(t_1 + j)).$$ \hspace{1cm} (52)

From (51) and Lemma 7, we obtain $|s(y(t_1 + j))| \leq \frac{1}{2}\Delta(t_1)$ for $j = 0, 1, ..., n - m - 1$. With (52), we have

$$|y(t_1 + n - m + j) - y^*(t_1 + n - m + j)| \leq c\Delta(t_1)$$ \hspace{1cm} (53)

which implies that

$$|y(t_1 + n - m + j)| \leq c\Delta(t_1) + |y^*(t_1 + n - m + j)|.$$ \hspace{1cm} (54)

Thus, from Lemma 6, we obtain

$$|y(t_1 + n - m + j)| \leq \Delta(t_1) \left(M - \frac{1}{2}\right)$$ \hspace{1cm} (55)

which implies that $q(y(t_1 + n - m + j))$, $j = 0, 1, ..., n - m - 1$, all do not saturate. Repeating the above procedure, we can further verify that $q(y(t_1 + n - m + j))$ for all $j = n - m - 1$ do not saturate. Thus, we conclude that, if $\Delta(t)$ is chosen as $\Delta(t_1)$ in (48) for all $t \geq t_1$, $q(y)$ will never saturate and $e(t)$ satisfies that

$$|e(t)| \leq c_0\Delta(t_1), \forall t \geq t_1 + n - m.$$ \hspace{1cm} (56)

Thus, there exists a well-defined number $t_2$ as

$$t_2 \triangleq \min\{t_2 \geq t_1 + 1 : |q(y(t))| \leq \frac{c\Delta(t_1) + |y^*(t)|}{\Delta(t_1)} + \frac{1}{2}\}.$$ \hspace{1cm} (57)

When $t = t_2$, we change the control law to (32) with $\Delta(t) = \Delta(t_2)$, where $\Delta(t_2)$ is defined in (48). Then, recalling the input-output delay $n - m$, we derive that $y(t + j)$, $j = 0, 1, ..., n - m - 1$, are still controlled by (32) with $\Delta(t) = \Delta(t_1)$. Thus, $y(t + j)$ satisfies

$$|y(t_2 + j)| \leq c\Delta(t_1) + |y^*(t_2 + j)|$$ \hspace{1cm} (58)

for $j = 0, 1, ..., n - m - 1$. Therefore,

$$|y(t_2 + j)| \leq \gamma\Delta(t_1) \left(\frac{c}{\gamma} + \frac{|y^*(t_2 + j)|}{\gamma\Delta(t_1)}\right)$$ \hspace{1cm} (59)

which follows from (29) that

$$|y(t_2 + j)| \leq \Delta(t_2) \left(M - \frac{1}{2}\right).$$ \hspace{1cm} (60)

Thus, $q(y(t_2 + j))$, $j = 0, 1, ..., n - m - 1$, all do not saturate, based on which we can further verify that $q(y(t_2 + j))$ does not saturate for all $j \geq n - m$. Hence, we conclude that, if the control law is chosen as (11) with $\Delta(t) = \gamma\Delta(t_1)$ for $t \geq t_2$, $q(y)$ will never saturate and $e(t)$ satisfies that

$$|e(t)| \leq \gamma c\Delta(t_1), \forall t \geq t_2 + n - m.$$ \hspace{1cm} (61)

Repeating the above procedure, we define $t_2, t_3, ..., t_N$ of the form (34), and obtain a sequence $\{\Delta(t_i)\}_{1 \leq i \leq N}$. Then, it can be verified that the control law (32) with $\Delta(t)$ in (48) ensures that $e(t)$ satisfies (49). Since $y^*(t) \in L^\infty$, we have $y(t) \in L^\infty$. Under Assumption (A1), the boundedness of $y(t)$ implies that of $u(t)$.

$$\nabla$$

5 Simulation study

In this section, a representative example is given to illustrate the design procedure and the validity of the theoretical results.

Simulation model. Consider the following system

$$A_s(z)y(t) = k_psB_s(z)u(t),$$ \hspace{1cm} (62)

where $k_ps = 1$, and

$$A_s(z) = (z + 1)(z - 2) \left(z + \frac{1}{2}\right),$$ \hspace{1cm} (63)

$$B_s(z) = z \left(z + \frac{1}{2}\right).$$ \hspace{1cm} (64)

It follows from (63) and (64) that $A_s(z)$ is unstable and $B_s(z)$ is stable. Moreover, $A_s(z)$ and $B_s(z)$ have a common factor $z + \frac{1}{2}$ which is corresponding to the uncontrollable or unobservable mode of the state-space form of
In a word, the controlled plant is minimum-phase, but not completely controllable or observable.

Specification of $\theta_1^*$, $\theta_2^*$ and $y^*(t)$. A key step for constructing the quantized-output feedback MRC law (32) is to specify $\theta_1^* \in \mathbb{R}^2$ and $\theta_2^* \in \mathbb{R}^3$. Note that $\theta_1^*$ and $\theta_2^*$ in (32) are the same with those in the standard output feedback MRC law (8). Thus, following the procedure of deriving $\theta_1^*$ and $\theta_2^*$ for standard output feedback MRC law of general discrete-time LTI systems in Tao (2003), we calculate $\theta_1^*$ and $\theta_2^*$ as

$$
\theta_1^* = \left[ 0, -1 \right]^T, \quad \theta_2^* = \left[ -1, -\frac{5}{2}, -\frac{1}{2} \right]^T.
$$

Moreover, $\phi_1(t)$ and $\phi_2(t)$ are specified as

$$
\phi_1(t) = \omega_1(z)u(t), \quad \phi_1(z) = [z^{-2}, z^{-1}]^T, 
$$

$$
\phi_2(t) = \omega_2(z)y(t), \quad \phi_2(z) = [z^{-2}, z^{-1}, 1]^T.
$$

The reference output signal is chosen as

$$
y^*(t) = \frac{1}{2} \sin(t) - \frac{1}{3} \cos(0.5t).
$$

One can verify that exact output tracking can be achieved by applying the standard MRC law (8) with $\theta_1^*$ and $\theta_2^*$ in (65) and $\phi_1(t)$ and $\phi_2(t)$ in (66)-(67) to the system model (62).

Quantized-output feedback MRC law. From (27) and (63), we choose $\lambda = 3$. From (30) and (65), we obtain that $c = 4$. With (68), we choose $d = 1$. Based on (32) and (33), the constant parameters $c_0$, $M$, $k$ are chosen as $c_0 = 1$, $M = 3 \times 10^3 + 1$, $k = 1$. Moreover, we choose $\gamma = \frac{1}{4}$. Then, from (29), we determine $N = 13$. The control law in the simulation with $t_0 = 0$ is specified as

$$
u(t) =
\begin{cases}
0, & t \in [0, t_1), \\
\frac{1}{2}u(t-1) + r(t) + \Delta(t_i) & \text{if } t \in [t_i, t_{i+1}), \\
(-z^{-2} - \frac{5}{2}z^{-1} - \frac{1}{2})q(y(t)) & \text{if } i = 1, \ldots, N-1, \\
\frac{1}{2}u(t-1) + r(t) + \Delta(t_N) & \text{if } t \in [t_N, \infty),
\end{cases}
$$

where $\Delta(t) = 3^i$ for $t \in [0, t_1)$, and

$$
r(t) = \frac{1}{2} \sin(t + 1) - \frac{1}{3} \cos(0.5t + 0.5), 
$$

$$
\Delta(t_i) = \frac{3^{t_i}}{2^{t_i - 1}}, \quad i = 1, 2, \ldots, 13.
$$

with $t_1 = \min \{ t_1 \geq 1 : |q(y(t))| \leq 3 \times 10^3 \}$ and

$$
t_i = \min \{ t_i \geq t_{i-1} + 1 : |q(y(t))| \leq \frac{\Delta(t_{i-1}) + |y^*(t)|}{\Delta(t_{i-1})} + \frac{1}{2} \}
$$

From (4), $q(y)$ is specified as

$$
q(y) =
\begin{cases}
3001, & \text{if } y > (3000 + \frac{1}{3})\Delta, \\
\frac{3001 + 1}{2} \Delta & \text{if } -(3001 + \frac{1}{3})\Delta < y \leq (3001 + \frac{1}{3})\Delta, \\
-3001, & \text{if } y \leq -(3001 + \frac{1}{3})\Delta.
\end{cases}
$$

Simulation results. To show the proposed control algorithm independent of the initial conditions, we choose $y(0) = 9000$ which is much larger than the saturated value $M$ of the quantizer $q(y)$.

Fig. 1 shows the response of the system output $y(t)$ versus the reference output $y^*(t)$ with Fig. 2 showing the tracking performance when $t \geq 10$. From Fig. 1 and Fig. 2, we see that a satisfactory tracking is achieved when $t$ is larger than about 18. Fig. 3 shows the response of the
quantized-output feedback MRC law (69). Moreover, we present the response of the sensitivity \( \Delta(t) \) in Fig. 4. In particular, the changes of \( t_i, i = 1, 2, ..., 13 \), in (71) can be clearly obtained from Fig. 4. Specifically, we see from Fig. 4 that \( t_1 = 3, t_2 = 6, t_3 = 8, t_4 = 9, t_5 = 10, t_6 = 11, t_7 = 12, t_8 = 13, t_9 = 14, t_{10} = 15, t_{11} = 16, t_{12} = 17, t_{13} = 18 \). By the way, \( \Delta(t) = \frac{3^3}{2^9} = 0.00659 \) for \( t \geq 18 \). Finally, we present the response of the quantizer \( q(y(t)) \) in Fig. 5, in which \( q(y(t)) \) is no longer saturated when \( t \geq 3 \). The evolution of \( \Delta(t) \) and \( q(y(t)) \) exactly matches the theoretical results.

In summary, the simulation results for the system model (62) have not only verified the validity of the proposed method, but also verified the non-dependence on the initial conditions. Moreover, recalling that \( A_s(z) \) and \( B_s(z) \) are not corprime, the simulation results also verified the non-dependence of the proposed method on the controllable or observable condition.

**Fig. 4. Response of \( \Delta(t) \) in (71).**

**Fig. 5. Response of the quantizer \( q(y(t)) \) in (73).**

### 6 Concluding remarks

This paper has given a positive answer to the question: whether a quantized-output feedback version of the standard MRC law (32) is still effective for control of general discrete-time LTI systems without any additional design conditions. The quantized-output feedback MRC law is analytically constructed, and only replies on the minimum-phase condition.

It would be interesting to further consider the following problems: (i) if the coefficients of \( A(z) \) and \( B(z) \) are unknown, how to realize adaptive control based on the proposed control scheme in this paper? (ii) if \( B(z) \) is not stable, that is, the system (1) is non-minimum phase, whether a quantized-output feedback version of the well-known pole placement method in Tao (2003) is still effective to achieve closed-loop stability and output tracking? (iii) whether the proposed method in this paper can be extended to the nonlinear systems case? These deserve further investigation.

### References


